Knotted streamtubes in incompressible hydrodynamical flow and a restricted conserved quantity

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For certain families of fluid flow, a conserved quantity—stream helicity—has been conjectured. Using examples of linked and knotted streamtubes, it has been shown that stream helicity does, in certain cases, entertain itself with a very precise topological meaning, viz., a measure of the degree of knottedness or linkage of stream tubes. As a consequence, stream helicity may emerge as a robust topological invariant.

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I. INTRODUCTION

Lord Kelvin (who along with Helmholtz pioneered the subject of vortex motion) recognized in the late 19th century that in an inviscid and barotropic fluid being acted upon by irrotational body forces, any linkage or any knottedness in the vorticity field at any earlier time should remain conserved at all later times. After almost 100 years, Moreau [1] and later Moffatt [2] established an invariant known as helicity which is of topological character and encompasses Kelvin's insight. The stark analogy between vorticity ($\vec{\omega}$) in ordinary fluid dynamics and magnetic field ($\vec{B}$) in magnetohydrodynamics (MHD) prompted Moffatt to give similar topological interpretations to magnetic helicity and cross helicity (which, by the way, measures the degree of "mutual" knottedness of the two fields $\vec{\omega}$ and $\vec{B}$). Hence, researchers were able to effectively connect the two very rich fields—viz., topology and fluid dynamics—and excited a lot of interest in this direction. But what Lord Kelvin had missed was the possible existence of knotted streamtubes in steady Euler flows, a fact very logically speculated on by Moffatt [3]. Not much has been done on that. Here, in this paper, inspired by the works of Moffatt, we shall introduce in Sec. II a quantity which we shall call "stream helicity" (S) in inviscid and incompressible fluid being forced by irrotational body forces. It will be shown that one may conjecture that the stream helicity is a conserved quantity under certain restrictions which are not, of course, very rare in practice. In Sec. III, we shall note how this conserved quantity can have a very sound topological meaning for at least some kinds of flows and, hence, how stream helicity can be raised to the status of a topological invariant for linked and knotted streamtubes.

II. STREAM HELICITY

Let us start with the Euler equation [Eq. (1)] for three-dimensional, inviscid, incompressible fluid being acted upon by irrotational body forces. P used in the equation includes the effect of such forces also. Since the fluid is incompressible—i.e., the density is constant—we take the density to be unity for convenience:

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}P. \tag{1}$$

Incompressibility yields for the velocity field $\vec{u}$

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{2}$$

which helps in defining the vector potential $\vec{\xi}$ for the velocity field as follows:

$$\vec{u} = \vec{\nabla}\times\vec{\xi}. \tag{3}$$

Obviously, $\vec{\xi}$ is not unique, for a term $\vec{\nabla}\lambda$, λ being a scalar field, can always be added to it, keeping $\vec{u}$ unchanged. We shall come back to this issue in the right place. For now, let us put relation (3) into Eq. (1) to get:

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\xi}) + (\vec{u}\cdot\vec{\nabla})(\vec{\nabla}\times\vec{\xi}) = -\vec{\nabla}P. \tag{4}$$

But we have

$$\begin{aligned}[\vec{\nabla}\times(\vec{u}\cdot\vec{\nabla})\vec{\xi}]_i &= \epsilon_{ijk}\partial_j(u_l\partial_l\xi_k)\\ &= \epsilon_{ijk}(\partial_j u_l)(\partial_l\xi_k) + \epsilon_{ijk}u_l\partial_j\partial_l\xi_k\\ &= \epsilon_{ijk}(\partial_j u_l)(\partial_l\xi_k) + [(\vec{u}\cdot\vec{\nabla})(\vec{\nabla}\times\vec{\xi})]_i. \end{aligned} \tag{5}$$

Using relation (5) in Eq. (4) we get

$$\left[\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\xi}) + \vec{\nabla}\times\{(\vec{u}\cdot\vec{\nabla})\vec{\xi}\}\right]_i = \epsilon_{ijk}(\partial_j u_l)(\partial_l\xi_k) - \partial_i P$$

$$\Rightarrow \frac{\partial}{\partial t}\vec{\xi} + (\vec{u}\cdot\vec{\nabla})\vec{\xi} = \text{curl}^{-1}\vec{\eta}, \tag{6}$$

where $\vec{\eta}$ is defined as

$$\eta_i \equiv \epsilon_{ijk}(\partial_j u_l)(\partial_l\xi_k) - \partial_i P. \tag{7}$$

Now, let us define stream helicity (S) as

$$S \equiv \int_V \vec{\xi}\cdot\vec{u}\, d^3x, \tag{8}$$

where V is a volume occupied by the fluid. At this point let us ponder over the aforementioned nonuniqueness of the vector potential [4]. For a smooth discussion's sake, we assume for the time being that the volume is simply connected. Suppose $\xi_i \to \xi_i + \partial_i\lambda$; then, from the definition (8) of stream

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helicity we can find the change δS in S to be

$$\delta S = \int_V \vec{\nabla}\lambda \cdot \vec{u}\, d^3x = \oint_{\partial V} \lambda \vec{u} \cdot \hat{n}\, d^2x, \tag{9}$$

where $\hat{n}$ is the unit vector perpendicular to the infinitesimal surface element d^2x and we have used relation (2) and Gauss divergence theorem. Relation (9) amounts to saying that the stream helicity will be gauge invariant in case the surface ∂V bounding V is the surface made up of stream lines—i.e., $\vec{u}\cdot\hat{n}=0$ on ∂V. This condition for gauge invariance is rather strong because if $\vec{u}\cdot\hat{n}\neq 0$ on ∂V, then one cannot seek refuge in the Coulomb gauge for it is too loosely defined inside V with no information about the outside field whatsoever. More starkly, it means that different solenoidal vector potentials inside V can correspond to Coulomb potentials of fields which have different structures outside V. Now if we relax the condition that V be simply connected, then the line integrals of $\vec{\xi}$ about the "holes" in the possibly multiply connected region have to be specified in order to have gauge-invariant stream helicity within ∂V on which $\vec{u}\cdot\hat{n}=0$.

In the passing, one may note that $\vec{u}\cdot\hat{n}=0$ on ∂V is clearly only a sufficient condition, and not necessary. For δS to be 0, from Eq. (9), only the total surface integral needs to be zero, and so $\vec{u}\cdot\hat{n}$ could be nonzero, but could have positive and negative contributions on the surface, while still leading to the surface integral being zero. Thus, S might in fact be conserved even when $\vec{u}\cdot\hat{n}\neq 0$ on the surface.

We now wish to demonstrate that under certain restrictions this quantity is in fact conserved. So we take total derivative of S with respect to time to get

$$\frac{dS}{dt} = \int \frac{D}{Dt}(\vec{\xi}\cdot\vec{u})d^3x$$

$$\Rightarrow \frac{dS}{dt} = \int \vec{\xi}\cdot(-\vec{\nabla}P)d^3x + \int \vec{u}\cdot(\mathrm{curl}^{-1}\vec{\eta})d^3x, \tag{10}$$

where D/Dt is the material derivative with respect to time and it basically is shorthand for $\partial/\partial t + \vec{u}\cdot\vec{\nabla}$. Again, simple vector algebra suggests

$$(\vec{\nabla}\times\vec{\xi})\cdot(\mathrm{curl}^{-1}\vec{\eta}) = \vec{\nabla}\cdot(\vec{\xi}\times\mathrm{curl}^{-1}\vec{\eta}) + \vec{\eta}\cdot\vec{\xi}. \tag{11}$$

With relation (3) in mind, inserting relation (11) into Eq. (10), we have the following:

$$\frac{dS}{dt} = -2\int \vec{\xi}\cdot\vec{\nabla}P d^3x + \int \vec{\nabla}\cdot(\vec{\xi}\times\mathrm{curl}^{-1}\vec{\eta})d^3x + \int \xi_i\partial_l(\epsilon_{ijk}\xi_k\partial_j u_l)d^3x, \tag{12}$$

where Eq. (2) has been used. If all the terms on the right-hand side (RHS) of Eq. (12) vanish, then one may set

$$\frac{dS}{dt} = 0 \tag{13}$$

and say that stream helicity is a conserved quantity.

The first two terms on the RHS of Eq. (12) can be changed to integration over the surface which bounds the volume V in question (the surface will obviously extend to infinity if the fluid is unbounded) using the Gauss divergence theorem, and so if $\vec{\xi}$ decays fast enough to go to zero on the bounding surface, then these two term vanish; there may be other reasons for the terms to vanish as will be seen in the next section. Now, let us consider the third term. Although it seems to be very restrictive, one can see that in the following commonly occurring cases the integrand of this term trivially vanishes.

(i) The vector potential is one dimensional.

(ii) The vector potential has no dependence on the direction along the velocity field. (Other conditions given below are basically this condition's corollary.)

(a) $\vec{\xi}$ is two dimensional but has dependence only on the third direction.

(b) $\vec{\xi}$ is two dimensional with spatial variations only on the plane containing it.

(c) $\vec{\xi}$ is three dimensional but depends only on any one of the three independent directions.

One can see that such flows (for which the third term becomes zero) are very commonly found in any elementary textbooks on fluid mechanics. For example, in accordance with case (i), for the one-dimensional vector potential $\vec{\xi}=\hat{k}\Omega(x^2+y^2)/4$, the corresponding flow is $\vec{u}=-\hat{i}\Omega y/2+\hat{j}\Omega x/2$ which basically is the velocity field for three-dimensional fluid rotating counterclock about the z axis. As another instance, this time to go with case (ii) [ii(b), to be precise], is that of a uniform flow along the x direction: $\vec{u}=U\hat{i}$ which is generated by the vector potential $\vec{\xi}=-\hat{j}Uz/2+\hat{k}Uy/2$, which evidently is two dimensional with spatial variations only on the y-z plane containing it. A rather non-trivial case [as an example of case ii(a)] is for the flow: $\vec{u}=\hat{i}U\sin az+\hat{j}U\cos az$ (a is a constant) for which the vector potential is $\vec{\xi}=\hat{i}(U/a)\sin az+\hat{j}(U/a)\cos az$; readers must have noticed that this is just a variant of the more general flow—viz., ABC flow (see, e.g., Ref. [5]).

So we conjecture for the families of fluid flow for which the vector potential falls into the above set, and if eventually Eq. (13) holds, stream helicity is a conserved quantity. Also, for fluid flows for which it does not fall into the above set but the integration goes to zero for some reason or the other (which has not been investigated), S will remain conserved. In the next section we shall examine a scenario wherein stream helicity is conserved.

III. TOPOLOGICAL MEANING OF STREAM HELICITY

Now, we ask the question if it is possible to give stream helicity a topological meaning and, more importantly, can that topological meaning turn out to be a topological invariant. We shall see that the answer is in affirmative. To get both expectations met, one (other uninvestigated possibilities may also be there) of the ways seems to be the following: Consider two circular thin streamtubes which are singly linked

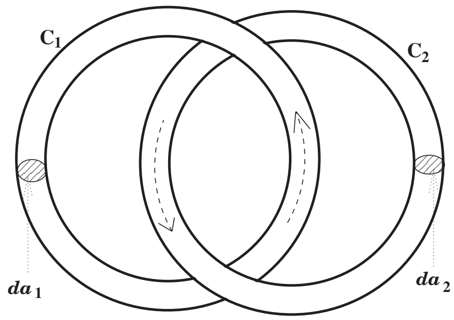

FIG. 1. Linked closed streamtubes. The tube and hence the stream lines inside are not twisted; i.e., the fluid inside the tube does not swirl. The directions of arrows are showing the direction of the stream lines filling the tubes.

and for the two tubes the "strengths" are V_{C_1} and V_{C_2}, respectively, where C_1 and C_2 denote axis circles of the corresponding tubes. By strengths we mean that $V_{C_1}=\vec{u}\cdot d\vec{a}_1$ and $V_{C_2}=\vec{u}\cdot d\vec{a}_2$ (see Fig. 1). Again, we assume that the velocity field we shall be dealing with is generated by a vector potential $\vec{\xi}$ which is a Beltrami field—i.e.,

$$\vec{u} = \vec{\nabla} \times \vec{\xi} = \alpha\vec{\xi}, \tag{14}$$

where α is a numerical constant. Moreover, suppose that of the conditions gathered in the previous section for $\vec{\xi}$, at least one is applicable, say, the second one, such that

$$(\vec{u} \cdot \vec{\nabla})\vec{\xi} = \vec{0}; \tag{15}$$

whether this is possible or not may be a valid question. One may derive "some" relief from the fact that if $\vec{\xi}$ is analogous to *ABC* flow [Gromeka (1881), Beltrami (1889)], then it does satisfy such a condition though unfortunately it may not sustain a linked structure of streamtubes. Then the streamtubes will be made up of stream lines which are coincident with the "flux lines" of the $\vec{\xi}$ field.

If we define the volume over which the integration is defined for the stream helicity to be the volume occupied by the linked structure only, then

$$S \equiv \int \vec{\xi}\cdot\vec{u}\, d^3x = \int\int\int_{\text{1st stream tube}} \vec{\xi}\cdot\vec{u}\, d^3x + \int\int\int_{\text{2nd stream tube}} \vec{\xi}\cdot\vec{u}\, d^3x$$

$$\Rightarrow S = V_{C_1}\int_{C_1} \vec{\xi}\cdot d\vec{l}_1 + V_{C_2}\int_{C_2} \vec{\xi}\cdot d\vec{l}_2$$

$$\Rightarrow S = V_{C_1}\int\int_{DC_1} \vec{u}\cdot d\vec{\sigma} + V_{C_2}\int\int_{DC_2} \vec{u}\cdot d\vec{\sigma}$$

$$\Rightarrow S = V_{C_1}V_{C_2} + V_{C_1}V_{C_2}$$

$$\Rightarrow S = 2V_{C_1}V_{C_2}, \tag{16}$$

where in the preceding steps we have used $\vec{u}d^3x \to V_{C_1}d\vec{l}_1$ and $V_{C_2}d\vec{l}_2$ on C_1 and C_2, respectively; DC_1 and DC_2 denote the area spanned by C_1 and C_2, respectively. Obviously, if the linking number is n and not 1 as in this case, one would easily generalize the result to

$$S = 2nV_{C_1}V_{C_2}, \tag{17}$$

which, being dependent on the mutual linking of streamtubes, is a topological quantity. One may rewrite Eq. (12) using the Gauss divergence theorem in the following form:

$$\frac{dS}{dt} = -2\int (\vec{\xi}\cdot\hat{n})Pd^2x + \int (\text{curl}^{-1}\vec{\eta})\cdot(\hat{n}\times\vec{\xi})d^2x + \int \xi_i\partial_l(\epsilon_{ijk}\xi_k\partial_j u_l)d^3x, \tag{18}$$

where $\hat{n}$ is the unit vector perpendicular to the surface at each point on the surface of the linked structure. The first term and the third terms of Eq. (18) are zero in this case by construction of the linked structure; so is the second term, but it needs a bit of manipulation as explained below.

First of all, we use Eq. (6) to rewrite the integrand of the second term of the relation (18) as

$$\left[\frac{\partial\vec{\xi}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{\xi}\right]\cdot(\hat{n}\times\vec{\xi}) = \epsilon_{ijk}n_j\xi_k\left[\frac{\partial\xi_i}{\partial t} + (u_l\partial_l)\xi_i\right]. \tag{19}$$

Now, if we consider the Frenet-Serret coordinate system $(\vec{T},\vec{N},\vec{B})$, then in the case we are considering $\vec{\xi}/|\vec{\xi}|=\vec{T}$ and $\hat{n}=\vec{N}$; obviously on the surface of the specific tube we are considering, at each point, the triad so that there $\xi_2=\xi_3=0$ and $n_1=n_3=0$ and hence due to the antisymmetry of ϵ_{ijk} we have

$$(\text{curl}^{-1}\eta)\cdot(\hat{n}\times\vec{\xi}) = 0. \tag{20}$$

So obviously we end up with the following relation:

$$\frac{dS}{dt} = 0. \tag{21}$$

Therefore, for incompressible, ideal, and conservatively forced fluid flow, in certain configurations, we can have a topological invariant—stream helicity—for linked structures of streamtubes.

So far, so good. So stream helicity does seem to make physical sense for linked two (or more) streamtubes. But what if a single streamtube is knotted? A single knotted streamtube must have an unavoidable twist of velocity field (which we hope, in this case also, may be derived from a Beltrami velocity vector potential and is of a similar kind as has been dealt with earlier in this paper) within the tube. How to deal with such a scenario has been discussed for knotted vortex filaments by Moffatt and Ricca [6]. We know when an arbitrary tame knot is viewed in a standard plane of projection with finite number of crossings, each of which is either positive or negative, it can be changed to an unknot (and ergo, subsequently continuously deformed to a circle) by switching the crossings for a finite number of times. (To remind the reader, a crossing is defined as positive or negative according as the overpass must be rotated counterclockwise or clockwise to bring it into coincidence with the underpass.) One may note that the resulting circle may be converted back to the original knot simple by performing the operations in reverse order. With this in mind, let us consider a tubular region with the circle as axis. The cross section of the tube is small, and over that the velocity of the field, which we suppose is filling the tube with strength V, is uniform; each stream line is, of course, a concentric circle to the circle serving as the axis. Now, let us transversely cut the tube somewhere and reconnect it back after giving it a twist through an angle $2\pi N$ (where N is an integer). This way we are introducing a stream helicity of magnitude NV^2. Then by introducing proper switching loops with similar strength, this construction may be changed to a knot with stream helicity

$$S = [N + 2(n_+ - n_-)]V^2, \tag{22}$$

where n_+ and n_- are, respectively, the number of positive and negative switches needed to create the knot whose stream helicity we are interested in. One may prove that $N+2(n_+ - n_-)$ is actually the linking number of any pair of stream lines in the knotted streamtube. It also is the self-linking number of the "framed" knot which is framed using the Frenet-Serret coordinate system [7]. A point to be noted is that for the kind of velocity field we are discussing S will remain conserved and hence emerges as a topological invariant, for, evidently, S depends on the topology of the knotted streamtube.

IV. CONCLUSION

To summarize, the existence of a conserved quantity—stream helicity—has been conjectured in fluid dynamics. By seeking a topological interpretation for it in certain configurations of linked and knotted streamtubes, the bridge between topology and fluid dynamics has been made even stronger. In addition, as a by-product, the seemingly nonphysical quantity—velocity vector potential—has given itself a sort of physical meaning by getting involved in measuring the degree of knottedness of streamtubes.

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[1] J. J. Moreau, C. R. Hebd. Seances Acad. Sci. **252**, 2810 (1961).
[2] H. K. Moffatt, J. Fluid Mech. **35**, 117 (1969).
[3] H. K. Moffatt, in *Mathematical Physics*, edited by A Fokas, A. Grigoryan, T. Kibble, and B. Zegarlinski (Imperial College Press, London, 2000), pp. 170–182.
[4] M. A. Berger and G. B. Field, J. Fluid Mech. **147**, 133 (1984).
[5] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics* (Springer-Verlag, New York, 1998).
[6] H. K. Moffatt and R. L. Ricca, Proc. R. Soc. London, Ser. A **439**, 411 (1992).
[7] W. F. Pohl, J. Fluid Mech. **17**, 975 (1968).